International Journal of Theoretical Physics, Vol. 27, No. 1, 1988

Loop Algebra of Lie Symmetries for a Short-Wave Equation

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Received August 13, 1987

It is demonstrated that Lie point symmetries associated with a nonlinear equation for short waves in three dimensions generate an infinite-dimensional Lie algebra—a loop Algebra. Classification of the independent sets of the subalgebra is done through the adjoint action of the corresponding generators. Different forms of similarity solutions are discussed.

1. INTRODUCTION

The importance of Lie symmetry in the analysis of nonlinear partial differential equations was realized long ago (Bluman and Cole, 1974). Initially all these algebras were finite-dimensional, but it was later seen that some symmetry generators do contain arbitrary functions of the time variable and it was also observed that a special class of dependence on this arbitrary function does lead to an infinite-dimensional Lie algebra which is isomorphic to a loop Algebra (Kac, 1985). Here we discuss this in three dimensions from the viewpoint of Lie symmetry and observe that the generators of point symmetry transformations do close on an infinite-dimensional algebra.

2. FORMULATION

The equation under consideration is written as

$$2K\phi_x + \phi_{yy} + 2\phi_{xt} - 2(x + \phi_x)\phi_{xx} = 0 \qquad (1)$$

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0020-7748/88/0100-0047$06.00/0

An important feature of this p.d.e. is that it is nonautonomous due to its explicit dependence on coordinate variable x. But one can easily observe that this nonautonomous equation is deducible from the following Lagrangian:

$$\mathscr{L} = (\phi_t\phi_x - \tfrac{1}{3}\phi_x^3 - x\phi_x^2 + \tfrac{1}{2}\phi_y^2)\exp[2(k+1)t] \tag{2}$$

where k is an arbitrary constant. The Lie point symmetries of this equation were originally studied by Khamitova (1982). Let us recapitulate some of the salient features of this analysis. The generator of transformation is

$$X = \eta(t, x, y, \phi, \phi_x, \phi_y)\partial/\partial\phi \tag{3}$$

where η is the following function:

$$\begin{aligned}\eta = {} & c_1\phi_t + (c_2x + \mu' y - p)\phi_x + (\tfrac{1}{3}c_2y - \mu)\phi_y - 2c_2\phi \\ & + (\mu'' + \mu')xy - (p' + p)x - \tfrac{1}{3}y^3 d_k\mu' + y^2 d_k p + \lambda y + \sigma\end{aligned} \tag{4}$$

where c_1 and c_2 are constant and μ, p, λ, and σ are arbitrary functions of t, and d_k is the operator

$$d_k = \frac{d^2}{dt^2} + (k+1)\frac{d}{dt} + k \tag{5}$$

Now by a basic theorem of Lie point transformation one can ascertain that the transformation generated by (η) is equivalent to the transformation through the following Lie operators:

$$\begin{aligned}
X_1 &= -\mu' y\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} + \left[xy(\mu'' + \mu') - \frac{y^3}{3}d_k(\mu')\right]\frac{\partial}{\partial\phi} \\
X_2 &= \tau\frac{\partial}{\partial x} + [y^2 d_k(\tau) - x(\tau' + \tau)]\frac{\partial}{\partial\phi} \\
X_3 &= \lambda y\frac{\partial}{\partial\phi} \\
X_4 &= \sigma\frac{\partial}{\partial\phi} \\
X_5 &= 9a\frac{\partial}{\partial t} + 3[(a' + 4a)p - (a'' - a')y^2]\frac{\partial}{\partial x} \\
&\quad + 6(a' - a)y\frac{\partial}{\partial y} - 3\left[(a' + 8a)\phi + \frac{x^2}{2}(a'' - a')\right. \\
&\quad \left. - (a'' - a')xy^2 + \frac{y^4}{6}(a'''' + 2a''' - a'' - 2a')\right]\frac{\partial}{\partial\phi}
\end{aligned} \tag{6}$$

We first compute the commutation rules between these generators; the result of such a computation can be displayed as:

$$\begin{aligned}
&[X_1(\mu), X_2(\tau)] = X_3[\mu'\tau' - \tau\mu'' + 2\mu d_k(\tau)] \\
&[X_1(\mu), X_3(\lambda)] = X_4[\mu\lambda] \\
&[X_1(\mu), X_4(\sigma)] = 0 \\
&[X_1(\mu), X_5(a)] = 3X_1[\mu'(2a' - 5a)] \\
&[X_2(\tau), X_3(\lambda)] = 0 \\
&[X_2(\tau), X_4(\sigma)] = 0 \\
&[X_2(\tau), X_5(a)] = 3X_2[\tau(a' - 4a) - 3\tau' a] \\
&[X_3(\lambda), X_4(\sigma)] = 0 \\
&[X_3(\lambda), X_5(a)] = -9X_3[\lambda(a' + 2a) + \lambda' a] \\
&[X_4(\sigma), X_5(a)] = -3X_4[\sigma(a' + 8a) + 3\sigma' a] \\
&[X_1(\mu_1), X_1(\mu_2)] = X_2(\mu_2\mu_1' - \mu_1\mu_2') \\
&[X_2(\tau_1), X_2(\tau_2)] = X_4(\tau_2\tau_1' - \tau_1\tau_2') \\
&[X_3(\lambda_1), X_3(\lambda_2)] = 0 \\
&[X_4(\gamma_1), X_4(\sigma_2)] = 0 \\
&[X_5(a_1), X_5(a_2)] = 9X_5(a_1 a_2' - a_1' a_2)
\end{aligned} \tag{7}$$

Equations (7) evidently show that the generators form a Lie algebra, but not the usual one, as the arguments occurring on the right-hand side are actually different. Since the functions μ, λ, σ, and a are all arbitrary and if we make an assumption that they are analytic functions in t, then equation (7) is nothing but a form of the loop or Virasoro algebra (Chau, 1983). For example let us set τ, σ, etc., each equal to t^m or t^n; then the commutation rules are

$$\begin{aligned}
&[X_1(t^m), X_3(t^n)] = X_4(t^{m+n}) \\
&[X_1(t^m), X_1(t^n)] = (m-n)X_2(t^{m+n-1}) \\
&[X_2(t^m), X_2(t^n)] = (m-n)X_4(t^{m+n-1}) \\
&[X_5(t^m), X_5(t^n)] = 9(n-m)X_5(t^{m+n-1}) \\
&[X_4(t^m), X_5(t^n)] = -3(3m+n)X_4(t^{m+n-1}) - 24X_4(t^{m+n}) \\
&[X_3(t^m), X_5(t^n)] = -9(m+n)X_3(t^{m+n-1}) - 18X_3(t^{m+n}) \\
&[X_2(t^m), X_5(t^n)] = 3(n-3m)X_2(t^{m+n-1}) - 12X_2(t^{m+n}) \\
&[X_1(t^m), X_5(t^n)] = 6mnX_1(t^{m+n-2}) - 15mX_1(t^{m+n-1})
\end{aligned} \tag{8}$$

all other commutators being zero. Before proceeding further, we try to identify the basic algebra underlying our loop algebra. If we denote the generator set by X_1^0, X_2^0, X_3^0, X_4^0, and X_5^0 obtained from equations (7) by setting the arbitrary functions equal to unity, then it immediately follows that (Jacobson, 1966)

$$r=\{x_2^0, X_3^0, X_4^0\}$$

generates a radical (Abelian subalgebra) and

$$SS=\{X_1^0, X_5^0\}$$

yields a semisimple part. Furthermore, we have the Levi decomposition;

$$\left.\begin{array}{rcl} [SS, SS] & C & SS \\ [SS, r] & C & r \\ [r, r] & C & r \end{array}\right\} \tag{9}$$

so that we can say

$$\{X_1^0, X_2^0, X_3^0, X_4^0, X_5^0\}=SS\{X_1^0, X_5^0\}\oplus r\{X_2^0, X_3^0, X_4^0\}$$

Now, $\{X_1^0, X_5^0\}$ is seen to be the generator of right translation algebra (Patera *et al.*, 1976). Thus, we may ascertain that our infinite-dimensional algebra is really generated by the direct product

$$(SS\{X_1^0, X_5^0\}\oplus r\{X_2^0, X_3^0, X_4^0\})\otimes L\{t, t^{-1}\}$$

where $L\{t, t^{-1}\}$ represents the space of Laurent polynomials in t. In the following we will demonstrate in detail how one can reduce these different Kac–Moody-type generators to simple Lie generators through the adjoint action of the corresponding group over the algebra.

3. ADJOINT ACTION AND REDUCTION

The problem noted in the previous section is actually the problem of constructing the so-called optimal system L_s that is the set of representatives of the class of S-dimensional subalgebra L_s, which are pairwise nonconjugate by the inner automorphism group. To proceed with the actual computation, we first observe that the adjoint action is usually given according to the formulas (Olver, 1985)

$$\mathrm{Ad}(\exp(\varepsilon v))\ W_0=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}-(\mathrm{ad}v)^n(W_0) \tag{10}$$

where $(\mathrm{ad}v)\,W_0=[V, W_0]$.

We enumerate the full adjoint operation results:

(5.1) $\mathrm{Ad}(\exp)[\varepsilon X_5(a)]\ X_1(\mu) = X_1(\mu_\varepsilon)$

$$\begin{aligned} \mu_\varepsilon &= \mu(t(t')) \\ t(t') &= \rho^{-1}[\rho(t) - 3\varepsilon] \\ \rho(t') &= -\int_{t_0}^{t'} \frac{dz}{3a(z)} \end{aligned} \tag{11}$$

(5.2) $\mathrm{Ad}(\exp)[\varepsilon X_5(a)])\ X_2(\tau) = X_2(\tau_\varepsilon)$

$$\tau_\varepsilon = \tau(t(t')) \left[\frac{a(t')}{a(t(t'))}\right]^{1/2} \exp\{\tfrac{4}{3}[t(t') - t']\}$$

so

$$\begin{aligned} X_2(\tau_\varepsilon) = \tau(t(t')) \left[\frac{a(t')}{a(t(t'))}\right]^{1/3} \exp\{\tfrac{4}{3}[t(t') - t'] \\ \times \frac{\partial}{\partial x'} + \left\{ y'^2 d_k \left[(t(t')) \left(\frac{a(t')}{a(t(t'))}\right)^{1/3} \right.\right. \\ \left.\left. \times \exp\{\tfrac{4}{3}[t(t') - t']\} \right] - x' \frac{d}{dt'} \tau_\varepsilon - x' \tau_\varepsilon \right\} \frac{\partial}{\partial \phi'} \end{aligned} \tag{12}$$

(5.3) $\mathrm{Ad}(\exp[\varepsilon X_5(a)])\ X_3(\lambda) = X_3(\lambda_\varepsilon)$

$$\lambda_\varepsilon = \lambda(t(t')) \frac{a(t(t'))}{a(t')} \exp\{2[t(t') - t']\}$$

so we can write explicitly

$$\begin{aligned} X_3(\lambda_\varepsilon) &= \lambda(t(t')) \frac{a(t(t'))}{a(t')} \exp\{2[t(t') - t']\} y' \frac{\partial}{\partial \phi'} \\ t'(t) &= (3\rho)^{-1}[3\rho(t) - 9\varepsilon] \end{aligned} \tag{13}$$

(5.4) $\mathrm{Ad}(\exp[\varepsilon X_5(a)])\ X_4(\sigma) = X_4(\sigma_\varepsilon)$

$$\sigma_\varepsilon = \sigma(t(t')) \left[\frac{a(t(t'))}{a(t')}\right]^{1/3} \exp \frac{8[t(t') - t']}{3}$$

so that

$$X_4(\sigma_\varepsilon)=\sigma(t(t'))\left[\frac{a(t(t'))}{a(t')}\right]^{1/3}\exp\frac{8[t(t')-t']}{3}\frac{\partial}{\partial\phi'} \tag{14}$$

(5.5) $\mathrm{Ad}(\exp)[\varepsilon X_5(a)]\ X_5(\alpha)=X_5(\alpha_\varepsilon)$ where

$$\alpha_\varepsilon=\alpha(t(t'))a(t')/a(t(t'))$$

with $t'(t)$ written as

$$t'(t)=(3\rho)^{-1}3[\rho(t)-9\varepsilon] \tag{15}$$

Now we know from Neuman's theorem that one can always choose a function $a(t)$ such that in equation (12) we can have

$$\tau(t(t'))\left[\frac{a(t')}{a(t(t'))}\right]^{1/3}\exp\frac{4[t(t')-t']}{3}=1$$

Then the generator $X_2(\tau_\varepsilon)$ reduces to

$$\tilde{X}_2(\tau_\varepsilon)=\frac{\partial}{\partial x'}+\{y^2K-x\}\frac{\partial}{\partial\phi'} \tag{16}$$

a generator of the finite-dimensional algebra. With another choice of $a(t)$ we can have, from equation (11),

$$\tilde{X}_1(\mu_\varepsilon)=\partial/\partial y$$

the translational operator in the y direction. Similarly, if we choose $a(t)$ so that

$$\lambda(t(t'))\frac{a(t(t'))}{a(t')}\exp\{2[t(t')-t']\}=1$$

then we observe that the generator X_3 reduces to

$$\tilde{X}_3=y'\partial/\partial\phi'$$

So with the help of the action of the adjoint operation of the group over the algebra we can reduce the generators to simple forms and hence can prove their mutual equivalence under the adjoint action.

4. SOLUTION GENERATED BY THE GENERATORS

Let us now try to build up the different types of solutions generated due to the invariance class generated by the various generators.

(1) X_4:

$$\phi'(t', x', y') = \phi(x, t, y) + \varepsilon\sigma(t)$$

(2) $X_4 + X_3$:

$$\phi'(t', x', y') = \phi(t, x, y) + \varepsilon[\sigma(t) + \lambda(t)y]$$

(3) $X_3 + X_4 + X_2$:

$$\phi'(t', x', y') = \phi[t, x - \varepsilon\tau(t), y] + \varepsilon[y' d_k(\tau(t') + y'\lambda(t') + \sigma(t')] \\ - \varepsilon[\tau'(t') + \tau(t')][X' + \tfrac{1}{2}\varepsilon\tau(t')]$$

Similar computations can be performed with the other combinations of generators. But we can also eliminate the parameter ε and express the corresponding nonlinear field ϕ as a known function of (x, t, y).

(4) $X_2 + X_3 + X_4 + X_1$:

$$t' = t$$

$$x' = x + \varepsilon[\tau(t) - \mu'(t)y - \tfrac{1}{2}\varepsilon\mu'(t)\mu(t)]$$

$$y' = y + \varepsilon\mu(t)$$

$$\begin{aligned}\phi'(t', x', y') = \phi[t', x' - \varepsilon\{\tau(t') - \mu'(t')y' \\ - \tfrac{1}{2}\varepsilon\mu'(t')\mu(t')\}, y' - \varepsilon\mu(t')] + [\mu''(t') + \mu'(t')] \\ \times[\varepsilon x'y' + \tfrac{1}{2}\varepsilon^2\{\mu(t')x' + \tau(t') - \mu'(t')y'\} \\ + \tfrac{1}{3}\varepsilon^3\{\mu(t')\tau(t') - \mu(t')\mu'(t')y' \\ - \tfrac{1}{2}\mu'(t')\mu(t')\} - \tfrac{1}{8}\varepsilon^4\mu'(t')\mu^2(t')] \\ - \tfrac{1}{3}[\varepsilon y'^3 + \tfrac{1}{4}\varepsilon^4\mu^3(t') + \tfrac{3}{2}\varepsilon^2 y'^2\mu(t') \\ + \varepsilon^3 y'\mu^2(t')]d_k(\mu'(t')) + [\varepsilon y'^2 \\ + \varepsilon^2 y'\mu(t') + \tfrac{1}{3}\varepsilon^3\mu^2(t')]d_k(\tau(t')) \\ -[\tau'(t') + \tau(t')][\varepsilon x' + \tfrac{1}{2}\varepsilon^2\{\tau(t') \\ - \mu'(t')y'\} - \tfrac{1}{6}\varepsilon^6\mu'(t')\mu(t')] \\ + \varepsilon[\lambda(t')y' + \sigma(t') + \tfrac{1}{2}\varepsilon\lambda(t')\mu(t')]\end{aligned}$$

(5) $X_2+X_3+X_4+X_5+X_1$:

$$t'(t)=\psi^{-1}[\psi(t)+9\varepsilon]$$

$$y'(t,y)=[y+G(t'(t),t)]\left[\frac{a(t'(t))e^{t}}{a(t)e^{t'(t)}}\right]^{2/3}$$

$$x'(t,x,y)=\left\{x+a^{1/3}(t)e^{-4t/3}\left[\int Q(t',t'(t),t,y)a^{-1/3}(t')\,e^{4t'/3}dt' - \int Q(t'(t),t,y)a^{-1/3}(t)\,e^{4t/3}\,dt\right]\right\}\left(\frac{a(t'(t))}{a(t)}\right)^{1/3}\left(\frac{e^{t}}{e^{t'(t)}}\right)^{4/3}$$

Set

$$\sigma(t'(t),t)=\frac{1}{9}\left(\frac{a(t)}{e^{t}}\right)^{2/3}4\int_{t}^{t'}\mu(S)[a(S)]^{-5/3}e^{2/3S}\,ds$$

and

$$\begin{aligned}
&Q(t',t'(t),y)\\
&=-\frac{1}{3}y^{2}\left(\frac{a(t'(t))e^{t}}{a(t)e^{t(t)}}\right)^{4/3}\left(\frac{a''(t')-a'(t')}{a(t')}\right)\\
&-\frac{y}{9}\left[\left(\frac{a(t'(t))e^{t}}{a(t)e^{t'(t)}}\right)^{2/3}\frac{\mu'(t')}{a(t')}\right.\\
&\left.+6\sigma(t'(t),t)\left(\frac{a(t'(t))e^{t}}{a(t)e^{t'(t)}}\right)^{4/3}\left(\frac{a''(t')-a'(t')}{a(t')}\right)\right]\\
&-\frac{1}{3}\sigma^{2}(t'(t),t)\left(\frac{a(t'(t))e^{t}}{a(t)e^{t'(t)}}\right)^{4/3}\left(\frac{a''(t')-a'(t')}{a(t')}\right)\\
&-\frac{1}{9}\sigma(t'(t),t)\left(\frac{a(t'(t))e^{t}}{a(t)e^{t'(t)}}\right)^{2/3}\left(\frac{\mu'(t')}{a(t')}\right)+\frac{1}{9}\frac{\tau(t')}{a(t')}
\end{aligned}$$

5. DISCUSSIONS

In the above analysis we considered the infinite-dimensional Lie algebra generated by the symmetry transformations of the equation of short waves. The corresponding reduction of the algebra via the adjoint action of the group was shown to be useful in classifying the various subalgebras. The solutions generated by these transformations were constructed in detail.

ACKNOWLEDGMENT

One of the authors (S.R.) is grateful to the DST (India) for a fellowship in a thrust area project.

REFERENCES

Bluman, G. W., and Cole, J. D. (1974). *Similarity Methods for Differential Equations*, Springer-Verlag.

Chau, L. L. (1983). In *Nonlinear Phenomena*, (CIFMO School), p. 110, Springer-Verlag.

Jacobson, N. (1966). *Lie Algebra*, New York.

Kac, V. (1985). *Infinite Dimensional Lie Algebra*, Birkhauser.

Khamitova, R. S. (1982). *Theoretical Mathematical Physics*, 244.

Olver, P. J. (1985). *On Application of Group Theory to Differential Equation*, Springer-Verlag.

Patera, J., Sharp, R. T. and Winternitz, P. (1976). *Journal Mathematical Physics*, 986.